Nonlocal abstract Stokes equations and applications

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In this talk, the Cauchy problem for the stationary and instationary nonlocal imcompressible abstract Stokes equations are considered. The equation involve the the convolution term and abstract operator in a Banach space E on leading part. The existence, uniqueness and coercive esimates in L^p spaces is derived. We can obtain a different classes Novier-Stokes equations by choosing the space E and the linear operator A which occur in a wide variety of physical systems. In application the existence, uniqueness and L^p-maximal regularity properties to solution of mixed problems for nonlocal degenerate Navier-Stokes equations and nonlocal Navier-Stokes equations with discontinuous coefficients are established.

We consider the Cauchy problem for the nonlocal Stokes equation

$$\frac{\partial u}{\partial t} - b * \Delta u + Au + \nabla \varphi = f\left(x, t\right), \; x \in \mathbb{R}^n, \; t \in \left(0, T\right), \tag{1.1}$$

$$\operatorname{div} u = 0, \; u\left(x, 0\right) = a\left(x\right), \tag{1.2}$$

where A is a linear operator in a Banach space E,

$$u = \left(u_1\left(x,t\right), u_2\left(x,t\right), ..., u_n\left(x,t\right)\right)$$

is an E-valued unknown solution

$$f = \left(f_1\left(x,t\right), f_2\left(x,t\right), ..., f_n\left(x,t\right)\right)$$

is given and $a = \left(a_1\left(x\right), a_2\left(x\right), ..., a_n\left(x\right)\right)$ is a initial date. Moreover,

$$b = b\left(x\right) = \left(b_1\left(x\right), b_2\left(x\right), ...b_n\left(x\right)\right), \; b * u = \left(b_1 * u_1, b_2 * u_2, ..., b_n * u_n\right),$$

$b_i * u_i$ denotes the convolution of the functions b_i, u_i defined by

$$b_i * u_i = \int\limits_{\mathbb{R}^n} b_i\left(x\right) u_i\left(x - \xi\right) d\xi$$

for smooth enough complex-valued functon b_i and E-valued function $u\left(x,t\right)$. Here, $\varphi = \varphi\left(x,t\right)$ is represent an E-valued unknown pressure. This problem is characterized with the presence of abstract operator A and the convolution term $b * \Delta u$. We obtain the well-posedeness of the problem $(1.1) - (1.2)$ in E-valued Bohner space. For $E = \mathbb{C}$, where $\mathbb{C}$ is the set of all complex numbers, and A is a positive constant q the problem $(1.1) - (1.2)$ is reduced to the following nonlocal Stokes type problem

$$\frac{\partial u}{\partial t} - b * \triangle u + qu + \nabla \varphi = f\left(x, t\right), \; \operatorname{div} u = 0, \tag{1.3}$$

$$u(x,0)=a(x)\,,\; x\in\mathbb{R}^n,\; t\in(0,T)\,. \tag{1.4}$$

Note that, the existence of weak or strong solutions and regularity properties for the classical Stokes problems extensively studied e.g. in [2, 4-8, 10, 12, 13, 18]. There is a lots of monograph on the solvability of the Cauchy problems for Stokes equations (see e.g. [1, 2,18] and further papers cited there). Solonnikov [12] proved that for every $f\in L^p\left(\Omega\times(0,T)\,;\mathbb{R}^3\right)=B(p)$, $p\in(1,\infty)$ the stationary Stokes problem

$$\frac{\partial u}{\partial t}-\triangle u+\nabla\varphi=f(x,t)\,,\;\; \operatorname{div}u=0,\;\; u\mid_{\partial\Omega}=0, \tag{1.5}$$

$$u(x,0)=0,\;\; x\in\Omega,\;\; t\in(0,T)$$

has a unique solution $(u,\nabla\varphi)$ so that

$$\left\|\frac{\partial u}{\partial t}\right\|_{B(p)}+\left\|\nabla^2 u\right\|_{B(p)}+\left\|\nabla\varphi\right\|_{B(p,q)}\leq C\left\|f\right\|_{B(p,q)}.$$

Giga and Sohr [5] improved the result for spaces with different exponents in space and time. Moreover, estimate obtained here, was global in time and used abstract parabolic semigroup theory. The estimate (1.5) allows to study the existence of solution and regularity properties of the corresponding Navier-Stokes problem (see e.g. [7]).

First all of, we consider the following nonlocal abstract differential equation (ADE) in whole space,

$$-b*\Delta u+(A+\lambda)\,u=f(x)\,,\, x\in\mathbb{R}^n, \tag{1.6}$$

where A is a linear operator in a Banach space E, $b=b(x)$ is a complex-valued function and λ is a complex parameter. We show the uniform maximal regularity properties of the problem (1.6), i.e. we prove that for all $f\in W^{m,q}(\mathbb{R}^n;E)$, $\lambda\in S_\psi$ problem (1.6) has a unique solution u that belongs to $W^{2+m,q}(\mathbb{R}^n;E(A)\,,E)$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{m+2}}\left\|b*\frac{\partial^i u}{\partial x_k^i}\right\|_{L^q(\mathbb{R}^n;E)}+\left\|Au\right\|_{L^q(\mathbb{R}^n;E)}\leq C\left\|f\right\|_{L^p(\mathbb{R}^n;E)},$$

where the constant C dependent only on A and E.

Then, we consider the stationary nonlocal abstract Stokes problem

$$-b*\triangle u+Au+\nabla\varphi=f(x)\,,\;\operatorname{div}u=0,\, x\in\mathbb{R}^n, \tag{1.7}$$

where $f=(f_1(x)\,,f_2(x)\,,...,f_n(x))$ is a data and $u=(u_1(x)\,,u_2(x)\,,...,u_n(x))$ is a solution. By applying the corresponding projection transformation P, the problem (1.7) can be reduced to problem

$$-Pb*\triangle u+Au=f(x)\,,\, x\in\mathbb{R}^n. \tag{1.8}$$

Let $L_{\sigma}^{q}\left(\mathbb{R}^{n};E\right)$ denotes the solenoidal space. Consider the operator in $L_{\sigma}^{q}\left(\mathbb{R}^{n};E\right)$ defined by

$$D\left(O_{q}\right)=\left(W_{\sigma}^{2,q}\left(\mathbb{R}^{n};E\left(A\right),E\right)\right)^{n}=\left\{u\in\left(W^{2,q}\left(\mathbb{R}^{n};E\left(A\right),E\right)\right)^{n},\ \operatorname{div}u=0\right\},$$

$$O_{q}u=-Pb\ast\triangle u+Au.$$

O_{q} is a nonlocal Stokes operator generated by problem (1.8). We prove here, that O_{q} is a sectorial operator in $\mathbb{X}_{q}=\left(L^{q}\left(\mathbb{R}^{n};E\right)\right)^{n}$ and also is a generator of an analytic semigroup in $\mathbb{X}_{q}$. Now, we consider the Cauchy problem for instationary nonlocal Stokes equation

$$\frac{\partial u}{\partial t}-b\ast\triangle u+Au+\nabla\varphi=f\left(x,t\right),\ x\in\mathbb{R}^{n},\ t\in\left(0,T\right), \tag{1.9}$$

$$\operatorname{div}u=0,\ u\left(x,0\right)=a$$

and we prove the $L^{p}\left(0,T;\mathbb{X}_{q}\right)$-well-possedeness of the problem (1.9).

We can obtain different classes of nonlocal Stokes equations by assigning a concrete space E and an appropriate operator A, that occur in applications. As a first example we can choose $E=L^{p_{1}}\left(0,1\right)$ and $A=A_{1}$ to be degenerate differential operator in $L^{p_{1}}\left(0,1\right)$ with nonlocal boundary conditions

$$D\left(A_{1}\right)=\left\{u\in W_{\gamma}^{[2],p_{1}}\left(0,1\right),\ \alpha_{k}u^{[\nu_{k}]}\left(0\right)+\beta_{k}u^{[\nu_{k}]}\left(1\right)=0,\ k=1,2\right\},$$

$$A_{1}u=b_{1}\left(y\right)u^{[2]}+b_{2}\left(y\right)u^{[1]},\ x\in\mathbb{R}^{n},\ y\in\left(0,1\right),\ \nu_{k}\in\left\{0,1\right\}, \tag{1.10}$$

where $u^{[i]}=D^{[i]}u=\left(y^{\gamma}\frac{d}{dy}\right)^{i}u$ for $0\leq\gamma<1-\frac{1}{p}$, $b_{1}=b_{1}\left(y\right)$ is a contınous function, $b_{2}=b_{2}\left(y\right)$ is a bounded function on $[0,1]$ for a.e. $x\in\mathbb{R}^{n}$, α_{k}, β_{k} are complex numbers, and $W_{\gamma}^{[2],p_{1}}\left(0,1\right)$ is a weighted Sobolev space defined by

$$W_{\gamma}^{[2],p_{1}}\left(0,1\right)=\{\ u:u\in L^{p_{1}}\left(0,1\right),\ u^{[2]}\in L^{p_{1}}\left(0,1\right),$$

$$\left\Vert u\right\Vert_{W_{\gamma}^{[2],p_{1}}}=\left\Vert u\right\Vert_{L^{p_{1}}}+\left\Vert u^{[2]}\right\Vert_{L^{p_{1}}}<\infty.$$

Let $\Pi=\mathbb{R}^{n}\times(0,1)\times(0,T)$. We obtain the $L^{\mathbf{p}}\left(\Pi\right)$-maximal regularity property of the nonlocal mixed problem for the following nonlocal Stokes equation

$$\frac{\partial u}{\partial t}-b\ast\triangle_{x}u+b_{1}\left(y\right)D_{y}^{[2]}u+b_{2}\left(y\right)D_{y}^{[1]}u+\nabla\varphi=f\left(x,y,t\right), \tag{1.11}$$

$$\operatorname{div}u=0,\ u\left(x,y,0\right)=a,\ x\in\mathbb{R}^{n},\ t\in\left(0,T\right).$$

$$\alpha_{k}u^{[\nu_{k}]}\left(x,0,t\right)+\beta_{k}u^{[\nu_{k}]}\left(x,1,t\right)=0,\ k=1,2, \tag{1.12}$$

$$\left(x,\ y,t\right)\in\Pi,\ u=u\left(x,y,t\right),$$

where the mixed $L^{\mathbf{p}}(\Pi)$-norm is defined as

$$\|f\|_{L^{\mathbf{p}}(\Pi)}=\left(\int\limits_{\mathbb{R}^n}\int\limits_0^T\left(\int\limits_0^1|f(x,y,t)|^{p_1}\,dy\right)^{\frac{p}{p_1}}dx\;dt\right)^{\frac{1}{p}}<\infty. \tag{1.13}$$

As a second example we can choose $E=L^{p_1}(0,1)$ and $A=A_2$ to be differential operator in $L^{p_1}(0,1)$ with VMO coefficients defined by

$$D(A_2)=\left\{u\in W^{2,p_1}(0,1)\,,\ \alpha_k u^{(\nu_k)}(0)+\beta_k u^{(\nu_k)}(1)=0,\ k=1,2\right\},$$

$$A_2u=b_1(y)\,u^{(2)}+b_2(y)\,u^{(1)},\ x\in\mathbb{R}^n,\ y\in(0,1)\,,\ \nu_k\in\{0,1\}\,. \tag{1.14}$$

After these, by applying our general results obtained here, we study the $L^{\mathbf{p}}(\Pi)$-well posedeness of mixed problems for the following nonlocal Stokes equations with VMO coefficients

$$\frac{\partial u}{\partial t}-b*\triangle_x u+b_1(y)\frac{\partial^2 u}{\partial x^2}+b_2(y)\frac{\partial u}{\partial x}+\nabla\varphi=f(x,y,t)\,, \tag{1.15}$$

$$\operatorname{div}u=0,\ u(x,y,0)=a,\ x\in\mathbb{R}^n,\ \ t\in(0,T)\,.$$

$$\alpha_k u^{(\nu_k)}(x,0,t)+\beta_k u^{(\nu_k)}(x,1,t)=0,\ k=1,2, \tag{1.16}$$

$$(x,\,y)\in\mathbb{R}^n\times(0,1)\,,\ t\in(0,T)\,,\ u=u(x,y,t)\,.$$

1. Notations and background

Let E be a Banach space and $L_p(\Omega;E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega\subset\mathbb{R}^n$ with the norm

$$\|f\|_{L_p}=\|f\|_{L_p(\Omega;E)}=\left(\int\limits_\Omega\|f(x)\|_E^p\,dx\right)^{\frac{1}{p}},1\leq p<\infty\ .$$

The Banach space E is called an UMD-space if the Hilbert operator

$$(Hf)(x)=\lim_{\varepsilon\to 0}\int\limits_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy$$

is bounded in $L^p(\mathbb{R},E)\,,\ p\in(1,\infty)$ (see. e.g. [3, 21]). UMD spaces include e.g. L^p, l_p spaces and Lorentz spaces L_{pq}, p, $q\in(1,\infty)$.

Let

$$S_{\psi}=\left\{ \lambda\in\mathbb{C},\ \left\vert \arg\lambda\right\vert \leq\varphi\cup\left\{ 0\right\} ,\ 0\leq\psi<\pi\right\} .$$

A closed linear operator A is said to be ψ-sectorial (or sectorial) in a Banach space E with bound $M>0$ if $D\left(A\right)$ and $R\left(A\right)$ are dense on E, $N\left(A\right)=\left\{0\right\}$ and

$$\left\Vert \left(A+\lambda I\right)^{-1}\right\Vert _{B(E)}\leq M\left\vert \lambda\right\vert ^{-1}$$

for any $\lambda\in S_{\phi}$, $0\leq\phi<\pi$, where I is the identity operator in E, $D\left(A\right)$ and $R\left(A\right)$ denote domain and range of the operator A, respectively.

It is known [19, §1.15.1] that there exist the fractional powers A^{θ} of a sectorial operator A. Let $E\left(A^{\theta}\right)$ denote the space $D\left(A^{\theta}\right)$ with norm

$$\left\Vert u\right\Vert _{E\left(A^{\theta}\right)}=\left(\left\Vert u\right\Vert ^{p}+\left\Vert A^{\theta}u\right\Vert ^{p}\right)^{\frac{1}{p}},\ 1\leq p<\infty,\ 0<\theta<\infty.$$

Let $\mathbb{N}$ denote the set of all natural numbers. A set $G\subset B\left(E_{1},E_{2}\right)$ is called R-bounded (see e.g. [3, § 2]) if there is a positive constant C such that for all $T_{1},T_{2},...,T_{m}\in G$ and $u_{1,}u_{2},...,u_{m}\in E_{1}$, $m\in\mathbb{N}$

$$\int\limits_{\Omega}\left\Vert \sum\limits_{j=1}^{m}r_{j}\left(y\right)T_{j}u_{j}\right\Vert _{E_{2}}dy\leq C\int\limits_{\Omega}\left\Vert \sum\limits_{j=1}^{m}r_{j}\left(y\right)u_{j}\right\Vert _{E_{1}}dy,$$

where $\left\{r_{j}\right\}$ is a sequence of independent symmetric $\left\{-1,1\right\}$-valued random variables on Ω. The smallest C for which the above estimate holds is called a R-bound of the collection G and denoted by $R\left(G\right).$

A set $G_{h}\subset B\left(E_{1},E_{2}\right)$ depending on parameter $h\in Q$ is called uniform R-bounded with respect to h if there is a constant C, independent on $h\in Q$, such that for all $T_{1}\left(h\right),T_{2}\left(h\right),...,T_{m}\left(h\right)\in G_{h}$ and $u_{1,}u_{2},...,u_{m}\in E_{1}$, $m\in\mathbb{N}$

$$\int\limits_{\Omega}\left\Vert \sum\limits_{j=1}^{m}r_{j}\left(y\right)T_{j}\left(h\right)u_{j}\right\Vert _{E_{2}}dy\leq C\int\limits_{\Omega}\left\Vert \sum\limits_{j=1}^{m}r_{j}\left(y\right)u_{j}\right\Vert _{E_{1}}dy.$$

It is implies that $\sup\limits_{h\in Q}R\left(G_{h}\right)\leq C.$

The ψ-sectorial operator A is said to be R-sectorial in a Banach space E if the set

$$L_{A}=\left\{ \xi\left(A+\xi\right)^{-1}:\ \xi\in S_{\psi}\right\} ,\ 0\leq\psi<\pi$$

is R-bounded.

The operator $A\left(\xi\right)$ is said to be ψ-sectorial in E uniformly with respect to ξ with bound $M>0$ if $D\left(A\left(\xi\right)\right)$ is independent on ξ, $D\left(A\left(\xi\right)\right)$ is dense in E and $\left\Vert \left(A\left(\xi\right)+\lambda\right)^{-1}\right\Vert \leq\frac{M}{1+\left\vert \lambda\right\vert }$ for all $\lambda\in S_{\psi}$, $0\leq\psi<\pi$, where M does not depend on ξ and λ.

Let E_0 and E be two Banach spaces and E_0 continuously and densely embedded into E. Let Ω be a measurable set in $\mathbb{R}^n$ and m be a positive integer. Let us consider the space $W^{p,m}(\Omega;E_0,E)$ consisting of all functions $u\in L^p(\Omega;E_0)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m}\in L^p(\Omega;E)$ with the norm

$$\|u\|_{W^{p,m}(\Omega;E_0,E)}=\|u\|_{L^p(\Omega;E_0)}+\sum_{k=1}^{n}\left\|\frac{\partial^m u}{\partial x_k^m}\right\|_{L^p(\Omega;E)}<\infty.$$

For $n=1$, $\Omega=(a,b)$, $a,b\in\mathbb{R}$ the space $W^{p,m}(\Omega;E_0,E)$ will be denoted by $W^{p,m}(a,b;E_0,E)$. For $E_0=E$ the space $W^{p,m}(\Omega;E_0,E)$ denotes by $W^{p,m}(\Omega;E)$.

BMO denotes the space of all complex-valued local integrable functions with the norm

$$\|f\|_*=\sup_B\int_B|f(x)-f_B|\,dx<\infty,$$

where B ranges in the class of the balls in $\mathbb{R}^n$ and f_B is the average $\frac{1}{|B|}\int_B f(x)\,dx$. For $f\in BMO$ and $r>0$ we set

$$\eta(r)=\sup_{\rho\leq r}\int_B|f(x)-f_B|_E\,dx,$$

where B ranges in the class of the balls with radius ρ.

We will say that a function $f\in BMO$ is in the space VMO if $\lim_{r\to+0}\eta(r)=0$. We will call $\eta(r)$ the VMO modulus of f. Here, $S(\mathbb{R}^n;E)$ denotes an E-valued Schwartz class, i.e. the space of all E-valued rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by seminorms. $S(\mathbb{R}^n;\mathbb{C})$ is denoted by $S(\mathbb{R}^n)$. Let $S'(\mathbb{R}^n;E)$ denote the space of all continuous linear operators from $S(\mathbb{R}^n)$ into E equipped with the bounded convergence topology. Recall $S(\mathbb{R}^n;E)$ is norm dense in $L^p(\mathbb{R}^n;E)$, when $1\leq p<\infty$.

Let F denotes the Fourier transform defined by

$$\hat{u}(\xi)=Fu=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-ix\xi}u(x)\,dx \text{ for } u\in S(\mathbb{R}^n;E) \text{ and } x,\ \xi\in\mathbb{R}^n.$$

Let $H^{q,s}(\mathbb{R}^n;E)$, $-\infty<s<\infty$ denotes E-valued Sobolev space of order s i.e

$$H^{q,s}(\mathbb{R}^n;E)=\Big\{u\in L^q(\mathbb{R}^n;E)\,,\ \|u\|_{H^{q,s}(\mathbb{R}^n;E)}=$$

$$\left.\left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}Fu\right\|_{L^q(\mathbb{R}^n;E)}<\infty\right\}.$$

Consider the space $H^{q,s}(\mathbb{R}^n;E_0,E)$ defined by

$$H^{q,s}(\mathbb{R}^n;E_0,E)=\{u\in H^{q,s}(\mathbb{R}^n;E)\cap L^q(\mathbb{R}^n;E_0)\,,$$

$$\|u\|_{H^{q,s}(\mathbb{R}^n;E_0,E)} = \|u\|_{L^q(\mathbb{R}^n;E_0)} + \|u\|_{H^{q,s}(\mathbb{R}^n;E)} < \infty \Big\}.$$

Sometimes we called it Sobolev-Lions space.

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α.

Moreover, for $u,\, \upsilon > 0$ the relations $u \lesssim \upsilon,\;\; u \approx \upsilon$ means that there exist positive constants $C,\, C_1,\, C_2$ independent on u and υ such that, respectively

$$u \leq C\upsilon,\;\; C_1\upsilon \leq u \leq C_2\upsilon.$$

From [19, Theorem A_0] we obtain:

Proposition $\mathbf{A}_1$. Let E be UMD space, $\beta = (\beta_1, \beta_2, ..., \beta_n)$, $D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} ... \partial x_n^{\beta_n}}$ and $1 < p \leq q \leq \infty$. Suppose $\Psi_h \in C^n\left(\mathbb{R}^n \backslash \{0\}; B(E)\right)$ and there is a positive constant K such that

$$\sup_{h \in Q} R\left(\left\{|\xi|^{|\beta| + n\left(\frac{1}{p} - \frac{1}{q}\right)} D^\beta \Psi_h(\xi) \colon \xi \in \mathbb{R}^n \backslash \{0\},\; \beta_k \in \{0,1\}\right\}\right) \leq K.$$

Then Ψ_h is a uniformly bounded collection of Fourier multiplier from $L^p(\mathbb{R}^n; E)$ to $L^q(\mathbb{R}^n; E)$.

2. Boundary value problems for abstract elliptic equations

BVPs for ADEs were studied e.g. in [1, 3, 11, 13-17, 22]. In this section, we will derive the maximal regularity properties of the problem (1.6).

Condition 2.1. Let E be an UMD space and A is an R-sectorial operator in E for $0 \leq \psi < \pi$. Assume $b \in L^1(\mathbb{R}^n)$, and $\hat{b} \in C^{(m)}(\mathbb{R}^n)$ such that

$$\sum_{k=1}^n \hat{b}_k(\xi)\, \xi_k^2 \in S_{\psi_1},\;\; \left|D^\beta \hat{b}_k(\xi)\right| \leq C_0 \text{ for } k = 1, 2, ..., n, \tag{2.1}$$

$$\psi_1 \leq \psi,\;\; |\beta|\,,\, m > 1 + \frac{n}{q},\; q \in (1,\infty) \text{ and for all } \xi \in \mathbb{R}^n.$$

Let $X_q = L^q(\mathbb{R}^n; E)$ and $Y^{q,2} = W^{q,2}(\mathbb{R}^n; E(A), E)$. We show here, the following result:

Theorem 2.1. Assume that the Condition 2.1 is satisfied and $q \in (1,\infty)$. Then for all $f \in X_q$, $\lambda \in S_\psi$ with $\psi_1 + \psi \in (0,\pi)$ problem (1.6) has a unique solution u that belongs to $Y^{q,2}$ and the following coercive uniform estimate holds

$$\sum_{k=1}^n \sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} \left\| b_k * \frac{\partial^i u}{\partial x_k^i} \right\|_{X_q} + \|Au\|_{X_q} \leq C\, \|f\|_{X_q}. \tag{2.2}$$

Indeed, by applying the Fourier transformation F in (1.6), we get

$$B\left(A,\xi,\lambda\right)\hat{u}\left(\xi\right)=\hat{f}\left(\xi\right),\ \xi\in\mathbb{R}^{n}, \tag{2.3}$$

where

$$B=B\left(A,\xi,\lambda\right)=A+\sum_{k=1}^{n}\hat{b}_{k}\left(\xi\right)\xi_{k}^{2}+\lambda.$$

By assumption $\lambda+\hat{b}\left(\xi\right)\left|\xi\right|^{2}\in S_{\psi}$, so the operator $B=B\left(A,\xi,\lambda\right)$ has bounded inverse in E for all $\xi\in\mathbb{R}^{n}$. Hence the equations has a solution

$$\hat{u}\left(\xi\right)=B^{-1}\left(A,\xi,\lambda\right)\hat{f}\left(\xi\right),$$

i.e. the exists a solution $u\left(x\right)$ of the equation (1.6) exspressed as

$$u\left(x\right)=F^{-1}\left[B^{-1}\left(A,\xi,\lambda\right)\hat{f}\left(\xi\right)\right]. \tag{2.4}$$

Let we now show the estimate (2.2). Indeed, in view of (2.4) the estimate (2.2) is equivalent to the following

$$\sum_{k=1}^{n}\sum_{i=0}^{2}\left|\lambda\right|^{1-\frac{i}{2}}\left\|F^{-1}\xi_{k}^{i}\hat{b}_{k}\left(\xi\right)B^{-1}\left(A,\xi,\lambda\right)\hat{f}\left(\xi\right)\right\|_{X_{q}}+ \tag{2.5}$$
$$\left\|AB^{-1}\left(A,\xi,\lambda\right)\hat{f}\left(\xi\right)\right\|_{X_{q}}\lesssim\left\|F^{-1}\hat{f}\left(\xi\right)\right\|_{X_{q}}.$$

To prove this, it is sufficient to show that the operator functions

$$\eta_{1}\left(\xi,\lambda\right)=\sum_{k=1}^{n}\sum_{i=0}^{2}\left|\lambda\right|^{1-\frac{i}{2}}\xi_{k}^{i}\hat{b}_{k}\left(\xi\right)B^{-1}\left(A,\xi,\lambda\right),\ \eta_{2}\left(\xi,\lambda\right)=AB^{-1}\left(A,\xi,\lambda\right)$$

are X_q-Fourier multipliers. Indeed, it is cleer to see that

$$\frac{\partial}{\partial\xi_{j}}\eta_{1}\left(\xi,\lambda\right)=-\sum_{k=1}^{n}\sum_{i=0}^{2}\left|\lambda\right|^{1-\frac{i}{2}}\xi_{k}^{i}\frac{\partial}{\partial\xi_{j}}B^{-1}\left(A,\xi,\lambda\right)=$$
$$-\sum_{k=1,k\neq j}^{n}\sum_{i=0}^{2}\left|\lambda\right|^{1-\frac{i}{2}}\xi_{k}^{i}\left[\frac{\partial}{\partial\xi_{j}}\hat{b}_{k}\left(\xi\right)+2\xi_{j}\right]B^{-2}\left(A,\xi,\lambda\right)- \tag{2.6}$$
$$\sum_{i=0}^{2}\left|\lambda\right|^{1-\frac{i}{2}}\left[i\xi_{j}^{i-1}B^{-1}\left(A,\xi,\lambda\right)+\left[\frac{\partial}{\partial\xi_{j}}\hat{b}_{k}\left(\xi\right)+2\xi_{j}\right]B^{-2}\left(A,\xi,\lambda\right)\right].$$

Really, by using the resolvent properties of operator A and assumption on $\hat{b}\left(\xi\right)$ and in view of (2.6) we have the following uniform in λ estimates

$$\left|\xi_{j}\right|\left\|\frac{\partial}{\partial\xi_{j}}\eta_{1}\left(\xi,\lambda\right)\right\|_{B(E)}\leq C_{1},\ j=1,2,...,n.$$

Then by differentiating the operator functions with respect other ξ_k, in a similar way get toe following uniform estimates

$$|\xi|^{|\alpha|} \|D^{\alpha}\eta_1(\xi,\lambda)\|_{B(E)} \leq C_1, \ \ |\xi|^{|\alpha|} \|D^{\alpha}\eta_2(\xi,\lambda)\|_{B(E)} \leq C_2, \tag{2.7}$$

for $|\alpha| > \frac{n}{p}$ with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_k \in \{0,1\}$, $\xi \in \mathbb{R}^n$, $\xi \neq 0$ and $\lambda \in S_{\psi_1}$.

Moreover, in view of (2.6) and due to R-sectoriality of the operator A, the sets

$$\left\{\xi_j \frac{\partial}{\partial \xi_j}\eta_1(\xi,\lambda), \, \xi \in \mathbb{R}^n \backslash \{0\}\right\},$$

$$\left\{\xi_j \frac{\partial}{\partial \xi_j}\eta_1(\xi,\lambda), \, \xi \in \mathbb{R}^n \backslash \{0\}, \, j = 1, 2, ..., n\right\}$$

are R-bounded. Then in view of the Kahane's contraction principle and from the product properties of the collection of R-bounded operators (see e.g. [3] Lemma 3.5, Proposition 3.4) we obtain

$$\sup_{\lambda \in S_\psi} R\left\{\xi^{\alpha} D^{\alpha}\eta_1(\xi,\lambda) \colon \, \xi \in \mathbb{R}^n \backslash \{0\}, \ |\alpha| > \frac{n}{p}\right\} \leq C_1, \tag{2.8}$$

$$\sup_{\lambda \in S_\psi} R\left\{\xi^{\alpha} D^{\alpha}\eta_2(\xi,\lambda) \colon \, \xi \in \mathbb{R}^n \backslash \{0\}, \ |\alpha| > \frac{n}{p}\right\} \leq C_2.$$

By Proposition A_1 from (2.8) we get that the operator-valued functions $\eta_1(\xi,\lambda)$ and $\eta_1(\xi,\lambda)$ are Fourier multipliers in X_q. Hence, we obtain the conclusion.

Consider a differential operator $Q = Q_q$ in X_q generated by problem (1.6), i.e.

$$D(Q) = Y^{q,2}, \, Qu = -b * \triangle u + Au.$$

Let $B_q = B(X_q)$. From Theorem 2.1 we obtain the following

Result 2.1. For $\lambda \in S_\psi$ there is a resolvent $(Q+\lambda)^{-1}$ satisfying the following uniform estimate

$$\sum_{k=1}^{n}\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\| \frac{\partial^i}{\partial x_k^i} b_k * (Q+\lambda)^{-1} \right\|_{B_q} + \left\| A(Q+\lambda)^{-1} \right\|_{B_q} \leq C.$$

In the following result we show the smoothness of the problem (2.1).

Theorem 2.2. Let the Condition 2.1 holds and m a positive integer $q \in (1,\infty)$. Then for all $f \in W^{q,m}(\mathbb{R}^n; E)$, $\lambda \in S_\psi$ problem (2.1) has a unique solution u that belongs to $Y^{q,2+m}$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \left\| b_k * \frac{\partial^i u}{\partial x_k^i} \right\|_{X_q} + \|Au\|_{X_q} \leq \tag{2.9}$$

$$C \left\| f \right\|_{W^{q,m}(\mathbb{R}^n;E)} .$$

Proof. From (2.3) it follows from the above expression that

$$\sum_{k=1}^{n} \sum_{i=0}^{m+2} \left|\lambda\right|^{1-\frac{i}{m+2}} \left\| b_k * \frac{\partial^i u}{\partial x_k^i} \right\|_{X_q} + \left\| Au \right\|_{X_q} = \tag{2.10}$$

$$\sum_{k=1}^{n} \sum_{i=0}^{m+2} \left|\lambda\right|^{1-\frac{i}{m+2}} \left\| F^{-1} \xi_k^i \hat{b}_k \left(\xi\right) B^{-1} \left(A, \xi, \lambda\right) \hat{f} \right\|_{X_q} +$$

$$\left\| F^{-1} A B^{-1} \left(A, \xi, \lambda\right) \hat{f} \right\|_{X_q} .$$

By reasoning as in the proof of Theorem 2.1 we show that the operator-functions

$$\Psi_\lambda \left(\xi\right) = A B^{-1} \left(A, \xi, \lambda\right) \left(1 + \sum_{k=1}^{n} \hat{b}_k \left(\xi\right) \xi_k^m \right)^{-1},$$

$$\sigma_\lambda \left(\xi\right) = \sum_{k=1}^{n} \sum_{i=0}^{m+2} \left|\lambda\right|^{1-\frac{i}{m+2}} \xi_k^i \left(1 + \sum_{k=1}^{n} \hat{b}_k \left(\xi\right) \xi_k^m \right)^{-1} B^{-1} \left(A, \xi, \lambda\right)$$

are uniform Fourier multipliers in X_q. Then by (2.9) we obtain the conclusion.

3. Nonlocal Stokes problem

In this section, we derive the maximal regularity properties of the stationary nonlocal abstract Stokes problem (1.7). Let we denote $H^{q,m} \left(\mathbb{R}^n; E\right)$ just by $H^{q,m}$. It is known that if E is a UMD space then $H^{q,m} = W^{q,m} \left(\mathbb{R}^n; E\right)$ for positive integer m (see e.g. [20, § 15]). Let $\mathbb{X}_q = \left(X_q\right)^n$ is class of E-valued system of function $f = \left(f_1 \left(x\right), f_2 \left(x\right), ..., f_n \left(x\right)\right)$ with norm

$$\left\| f \right\|_{\mathbb{X}_q} = \left(\sum_{i=1}^{n} \left\| f_i \right\|_{X_p}^q \right)^{\frac{1}{q}} .$$

Let $C_0^\infty \left(\mathbb{R}^n; E\right)$ denotes the set of all E-valued infinite many differentiable finite functions on $\mathbb{R}^n$. Moreover, $\mathbb{X}_{q\sigma} = L_\sigma^q \left(\mathbb{R}^n; E\right)$ denote the E-valued solenoidal space, i.e. closure of $\left(C_{0\sigma}^\infty \left(\mathbb{R}^n; E\right)\right)^n$ in $\mathbb{X}_q$, where

$$C_{0\sigma}^\infty \left(\mathbb{R}^n; E\right) = \left\{ u \in \left(C_0^\infty \left(\mathbb{R}^n; E\right)\right)^n , \ \operatorname{div} u = 0 \right\} .$$

Let $\mathbb{X}_{q,s} = \left(H^{q,s}\right)^n$ and $\mathbb{Y}^{q,s} = \left(Y^{q,s}\right)^n$. Consider the space

$$\mathbb{Y}_q = \{ u \in \mathbb{X}_q, \ \operatorname{div} u \in X_q,$$

$$\|u\|_{\mathbb{Y}_q} = \left(\|u\|_{\mathbb{X}_q}^q + \|\operatorname{div} u\|_{X_q}^q \right)^{\frac{1}{q}}.$$

$\mathbb{Y}_q$ becomes a Banach space with this norm. It is known that (see e.g. D. Fujiwara and H. Morimoto [6]) vector field $u \in \left(L^q\left(\mathbb{R}^n\right)\right)^n$ has a Helmholtz decomposition. In following theorem we generalize this result for E-valued function space X_q.

Theorem 3.1. Assume that the Condition 2.1 is satisfied and $q \in (1, \infty)$. Then $u \in \mathbb{X}_r$ has a Helmholtz decomposition, i.e. there exists a linear bounded projection operator P_q from $\mathbb{X}_q$ onto $\mathbb{X}_{q\sigma}$ with null space

$$N\left(P_q\right) = \left\{\nabla \varphi \in \mathbb{X}_q : \varphi \in L_{loc}^q\left(\mathbb{R}^n; E\right)\right\}.$$

In particular, for all $u \in \mathbb{X}_q$ has a unique decomposition $u = u_0 + \nabla \varphi$ with $u_0 \in \mathbb{X}_{q\sigma}$, $u_0 = P_q u$ so that

$$\|\nabla \varphi\|_{L^q(B;E)} + \|u_0\|_{\mathbb{X}_q} \leq C \|u\|_{\mathbb{X}_q}, \tag{3.1}$$

for any open ball $B \subset \mathbb{R}^n$. Moreover, $\left(\mathbb{X}_{q\sigma}\right)^* = \mathbb{X}_{q'\sigma}$, $\frac{1}{q} + \frac{1}{q'} = 1$.

In order to prove Theorem 3.1 we need some lemmas. Consider first, the problem

$$-b * \Delta u + (A + \lambda) u = f(x), \; x \in \mathbb{R}^n, \tag{3.2}$$

where $f = \left(f_1(x), f_2(x), ..., f_n(x)\right) \in X_q$ and $u = \left(u_1(x), u_2(x), ..., u_n(x)\right)$ is a solution of (3.2)

Lemma 3.1. Assume that the Condition 2.1 is satisfied**,** $q \in (1, \infty)$ and $-2 < s < \infty$. Then for all $f \in \mathbb{X}_{q,s}$, $\lambda \in S_\psi$ problem (3.2) has a unique solution $u \in Y^{q,2+s}$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n} \sum_{i=0}^{s+2} |\lambda|^{1-\frac{i}{s+2}} \left\| b_k * \frac{\partial^i u}{\partial x_k^i} \right\|_{\mathbb{X}_q} + \|Au\|_{\mathbb{X}_q} \lesssim \|f\|_{\mathbb{X}_{q,s}}. \tag{3.3}$$

Proof. By using the Fourier transform we see that the estimate (3.3) is equivalent to the following estimate

$$\left\| F^{-1} \left(1 + \sum_{k=1}^{n} \xi_k^2\right)^{\frac{s+2}{2}} B^{-1}(A, \xi, \lambda) \hat{f} \right\|_{\mathbb{X}_q} + \left\| F^{-1} A B^{-1}(A, \xi, \lambda) \hat{f} \right\|_{\mathbb{X}_q}$$

$$+ |\lambda| \left\| F^{-1} B^{-1}(A, \xi, \lambda) \hat{f} \right\|_{\mathbb{X}_q} \leq C \left\| F^{-1} \left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{f} \right\|_{\mathbb{X}_q}, \tag{3.4}$$

where $B(A, \xi, \lambda)$ is a operator function defined by (2.3).

In order to prove (3.4) it sufficient to show that the operator functions

$$\sum_{k=1}^{n} \sum_{i=0}^{s+2} |\lambda|^{1-\frac{i}{s+2}} \xi_k^i \hat{b}_k(\xi) B^{-1}(A, \xi, \lambda), \; A\left(1 + |\xi|^2\right)^{-\frac{s}{2}} B^{-1}(A, \xi, \lambda),$$

$$|\lambda|\left(1+|\xi|^{2}\right)^{-\frac{s}{2}}B^{-1}\left(A,\xi,\lambda\right)$$

are multipliers in X_q uniformly with respect to λ. This fact is derived as in the proof of Theorem 2.1, 2.2.

By reasoning as in [8, Lemma 2] we get

Lemma 3.2. $\left(C^{\infty}\left(\mathbb{R}^{n};E\right)\right)^{n}$ is dense in $\mathbb{Y}_p$.

Consider the problem

$$-b*\Delta\varphi+A\varphi+\lambda\varphi=\operatorname{div}f\left(x\right),\ x\in\mathbb{R}^{n}. \tag{3.5}$$

Here, we will assume that the following assumption is satisfied until the end of the Paragraph 3:

Assume that the Condition 2.1 is satisfied and $q\in(1,\infty)$.

From the Lemma 3.1 we obtain the following results:

Result 3.1. For all $f\in\mathbb{X}_q$, $\lambda\in S_\psi$ problem (3.5) has a unique solution $\varphi\in Y^{q,1}$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\left\|b_k*\frac{\partial^i\varphi}{\partial x_k^i}\right\|_{X_q}+\|A\varphi\|_{X_q}\lesssim\|\operatorname{div}f\|_{Y^{q,1}}.$$

Consider the operator $P=P_q$ defined by

$$D\left(P\right)=\mathbb{X}_q,\ Pf=f-\nabla\varphi,$$

where φ is a solution of the problem (3.5).

Result 3.2. $P_q\mathbb{X}_q$ is a closed subspace of X_q.

Lemma 3.3. The P_q is a linear bounded operator in $\mathbb{X}_q$ and $Pf=f$ if $\operatorname{div}f\left(x\right)=0$.

Proof. The linearity of the operator P is clear by construction. Moreover, by Result 3.1 we have

$$\|Pf\|_{\mathbb{X}_q}\leq\|f\|_{\mathbb{X}_q}+\|\nabla\varphi\|_{\mathbb{X}_q}\leq C\,\|f\|_{\mathbb{X}_q}. \tag{3.6}$$

If $\operatorname{div}f\left(x\right)=0$, then by Lemma 3.1 we get that $\varphi=0$, i.e. $Pf=f$.

Let E^*denotes the dual space of E.

Lemma 3.4. The conjugate of P_q is defined as $P_q^*=P_{q'}$, $\frac{1}{q}+\frac{1}{q'}=1$ and is bounded linear in $\left(L^{q'}\left(\mathbb{R}^n;E^*\right)\right)^n$.

Proof. It is known (see e.g. [1], [20]) that the dual space of $L^q\left(\mathbb{R}^n;E\right)$ is $L^{q'}\left(\mathbb{R}^n;E^*\right)$. Since $C_0^\infty\left(\mathbb{R}^n;E^*\right)$ is dense in $L^{q'}\left(\mathbb{R}^n;E^*\right)$ we have only to show $P_q^*\varphi=P_{q'}\varphi$ for any $\varphi\in C_0^\infty\left(\mathbb{R}^n;E^*\right)$. But this fact is deriving by reasoning as in [6, Lemma 5]. Moreover, by Lemma 3.4 the dual operator P_q^* is a bounded linear in $\left(L^{q'}\left(\mathbb{R}^n;E^*\right)\right)^n$.

Let

$$G_q=\left\{\nabla\varphi:\varphi\in Y^{q,1}\right\},\ \left(P_q\mathbb{X}_q\right)^{\perp}=$$

$$\left\{ f \in \left(L^{q'} \left(\mathbb{R}^n; E^* \right) \right)^n, \ \ \langle f, \upsilon \rangle = 0 \text{ for any } \upsilon \in P_q \mathbb{X}_q \right\}.$$

From Lemmas 3.3, 3.4 we obtain

Result 3.3. Any element $f \in \mathbb{X}_q$ uniquely can be expressed as sum of elements of $P_q \mathbb{X}_q$ and G_q.

In a similar way as Lemmas 6, 7 of [6] we obtain:

Lemma 3.5. The following are hold

$$\left(P_q \mathbb{X}_q \right)^{\perp} = G_{q'}, \ \frac{1}{q} + \frac{1}{q'} = 1,$$

$$\mathbb{X}_{q\sigma}^{\perp} = G_{q'}, \frac{1}{q} + \frac{1}{q'} = 1.$$

Now we are ready to prove the Theorem 3.1.

Proof of Theorem 3.1. From Lemma 3.5, we get that $\mathbb{X}_{q\sigma} = \left(P_q \mathbb{X}_q \right)^{\perp}$. Then, by construction of P_q we have $\mathbb{X}_q = \mathbb{X}_{q\sigma} \oplus G_q$. By lemmas 3.1, 3.3, we obtain the estimate (3.1). Moreover, by Result 3.2, G_q is a close subspace of $\mathbb{X}_q$. Then, it known that the dual space of quotient space $\mathbb{X}_q / G_q$ is $G_q^{\perp}$. In view of first assertion we have $\mathbb{X}_q / G_q = \mathbb{X}_{q\sigma}$ and by second equality of Lemma 3.5, we obtain the second assertion.

Theorem 3.2. For all $f \in \mathbb{X}_q$, $\varphi \in H^{1,p} \left(\mathbb{R}^n; E \right)$, $\lambda \in S_\psi$ problem (1.8) has a unique solution $u \in \mathbb{Y}^{q,2}$, $\varphi \in H^{1,p} \left(\mathbb{R}^n; E \right)$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\| b_k * \frac{\partial^i u}{\partial x_k^i} \right\|_{\mathbb{X}_q} + \|Au\|_{\mathbb{X}_q} + \|\nabla \varphi\|_{\mathbb{X}_q} \leq C \|f\|_{\mathbb{X}_q}. \tag{3.7}$$

Proof. By applying the operator P_q to equation (1.7) we get the Stokes problem (1.8). It is clear to see that

$$D(Q) = D(B) \cap \mathbb{X}_{q\sigma},$$

where Q is the Stokes operator generated by Stokes problem (1.8) and B is a operator in $\mathbb{X}_q$ generated by problem (1.8) for $\lambda = 0$, i.e.

$$D(B) = \mathbb{Y}^{q,2}, \ Bu = -b * \Delta u + Au.$$

Then by Lemma 3.1 we obtain the assertion.

Result 3.2. From Theorem 3.2 we get that Q is a sectorial operator in $\mathbb{X}_q$ and also generates a bounded holomorphic semigroup $S(t) = \exp(-Qt)$ for $t > 0$.

In a similar way as in [9] we show

Proposition 3.1. The following estimate holds

$$\|Q^{\alpha} S(t)\| \leq C t^{-\alpha}$$

for $\alpha \geq 0$ and $t > 0$.

Proof. From Theorem 3.2 we obtain that the operator Q is uniformly positive in $\mathbb{X}_q$, i.e. for $\lambda \in S_{\psi,\varkappa}$, $0 < \psi < \pi$ the following estimate holds

$$\left\|(Q+\lambda)^{-1}\right\| \leq \frac{M}{|\lambda|},$$

where the constant M is independent of λ. Then, by using Danford integral and operator calculus (see e.g in [3]) we obtain the assertion.

4. Well-possedeness of the instationary nonlocal Stokes problem

Let E_1 and E_2 be two Banach spaces. By $(E_1, E_2)_{\theta,p}$, $0 < \theta < 1, 1 \leq p \leq \infty$ will be denoted the interpolation spaces obtained from $\{E_1, E_2\}$ by the K-method [19, §1.3.2]. Let $\mathbb{Y}^{p,q,2} = W^{1,p}\left(0, T; \mathbb{Y}^{q,2}, \mathbb{X}_q\right)$.

In this section we will assume that the following assumption:

Assume that the Condition 2.1 is satisfied and $q \in (1, \infty)$**.**

We will show here, the well-posedeness of the problem $(1.1)-(1.2)$.

Theorem 4.1. For every $f \in L^p(0, T; \mathbb{X}_q) = B(p,q)$ and $a \in \left(\mathbb{Y}^{q,2}, \mathbb{X}_q\right)_{\frac{1}{p},p} = G(p,q)$, $p, q \in (1, \infty)$ there is a unique solution $u \in \mathbb{Y}^{p,q,2}$, $\varphi \in H^{1,p}(\mathbb{R}^n; E)$ and $a \in G(p,q)$ of the problem $(1.1)-(1.2)$ satisfying the following estimate

$$\left\|\frac{\partial u}{\partial t}\right\|_{B(p,q)} + \sum_{k=1}^{n} \left\|\frac{\partial^2 u}{\partial x_k^2}\right\|_{B(p,q)} + \|Au\|_{B(p,q)} + \|\nabla \varphi\|_{B(p,q)} \lesssim \tag{4.1}$$

$$\|f\|_{B(p,q)} + \|a\|_{G(p,q)}.$$

Proof. The problem $(1.1)-(1.2)$ can be expressed as the following abstract parabolic problem

$$\frac{du}{dt} + Qu = f(t), \; u(0) = a. \tag{4.2}$$

If we put $E = \mathbb{X}_q$ then by Proposition 3.1, operator Q is sectorial and generates bounded holomorphic semigroup in $\mathbb{X}_q$ for $q \in (1, \infty)$. Moreover, by using [14, Theorem 3.1] we get that the operator Q is R-sectorial in $\mathbb{X}_q$. Since $\mathbb{X}_q$ is an UMD space for $q \in (1, \infty)$, in a similar way as in [21, Theorem 4.2] we obtain that for all $f \in L^p(0, T; E)$ and $a \in B(p,q)$ there is a unique solution $u \in W^{1,p}(0, T, D(Q), E)$ of the problem (4.2) so that the following estimate holds

$$\left\|\frac{du}{dt}\right\|_{L^p(0,T;E)} + \|Qu\|_{L^p(0,T;E)} \leq C\left(\|f\|_{L^p(0,T;E)} + \|a\|_{(D(Q),\mathbb{X}_q)_{\frac{1}{p},p}}\right). \tag{4.3}$$

From the estimates (3.10) and (4.3) we obtain the assertion.

Remark 4.2. There are a lot of positive operators in concrete Banach spaces. Therefore, putting in (1.8) and (1.1) concrete Banach spaces instead of E and concrete positive differential, pseudo differential operators, or finite, infinite matrices, etc. instead of A by virtue of Theorem 3.2 and Theorem 4.1 we can obtained the maximal regularity properties of different class of stationary and instationary Stokes problems, respectively which occur in numerous physics and engineering problems.

5. Application

Consider the Stokes problem (1.11)$-$(1.12). Let $\mathbb{X}_{q,p_1} = \left(L^q\left(\mathbb{R}^n; L^{p_1}\left(0,1\right)\right)\right)^n$ is class of $L^{p_1}\left(0,1\right)$-valued system of function

$$f = \left(f_1\left(x\right), f_2\left(x\right), ..., f_n\left(x\right)\right)$$

with norm

$$\left\|f\right\|_{\mathbb{X}_{q,p_1}} = \left(\sum_{i=1}^{n} \left\|f_i\right\|_{L^{q,p_1}\left(\mathbb{R}^n \times (0,1)\right)}^q\right)^{\frac{1}{q}}.$$

Let $\mathbb{X}_{p,q,p_1} = L^p\left(0,T; \mathbb{X}_{q,p_1}\right)$ be Lebesue space with mixed norm and $\mathbb{Y}^{q,p_1,2} = \left(Y^{q,p_1,2}\right)^n$, where

$$Y^{q,p_1,2} = W^{q,2}\left(\mathbb{R}^n; W^{p_1,[2]}\left(0,1\right), L^{p_1}\left(0,1\right)\right).$$

Let A_1 differential operator in $L^{p_1}\left(0,1\right)$ defined by (1.10).

Theorem 5.1. Assume the second assumption of Condition 2.1 is satisfied. Then for every $f \in \mathbb{X}_{p,q,p_1}$ for $p, q, p_1 \in (1,\infty)$ there is a unique solution $u \in \mathbb{Y}^{q,p_1,2}$ and $\varphi \in H^{1,p}\left(\mathbb{R}^n; L^{p_1}\left(0,1\right)\right)$ of (1.11)-(1.12). Moreover, the following coercive estimate holds

$$\sum_{k=1}^{n} \left\|\frac{\partial^2 u}{\partial x_k^2}\right\|_{\mathbb{X}_{p,q,p_1}} + \left\|A_1 u\right\|_{\mathbb{X}_{p,q,p_1}} \lesssim \left\|f\right\|_{\mathbb{X}_{p,q,p_1}}. \tag{5.2}$$

Proof. By [14, Theorem 5.0] the operator A_1 is R-positive in $L^{p_1}\left(0,1\right)$. Therefore, all conditions of Theorem 3.2 are hold and we obtain the conclusion.

Consider now, the Stokes problem (1.15)$-$(1.16). Let $\mathbb{X}_{p,q,p_1} = L^p\left(0,T; \mathbb{X}_{q,p_1}\right)$ be Lebesue space with mixed norm and $\mathbb{Y}^{q,p_1,2} = \left(Y^{q,p_1,2}\right)^n$, where

$$Y^{q,p_1,2} = W^{q,2}\left(\mathbb{R}^n; W^{p_1,2}\left(0,1\right), L^{p_1}\left(0,1\right)\right).$$

Let $\omega_1 = \omega_1\left(y\right)$, $\omega_2 = \omega_2\left(y\right)$ be roots of equation $b_1\left(y\right)\omega^2 + 1 = 0$. Let

$$\nu\left(y\right) = \begin{vmatrix} \left(-\omega_1\right)^{m_1}\alpha_1 & \beta_1\omega_1^{m_1} \\ \left(-\omega_2\right)^{m_2}\alpha_2 & \beta_2\omega_2^{m_2} \end{vmatrix}, \ \mathbb{V}_{p,q} = \left(\mathbb{Y}^{q,p_1,2}, \mathbb{X}_{p,q,p_1}\right)_{\frac{1}{p},p}.$$

Let A_2 be differential operator in $L^{p_1}(0,1)$ defined by (1.14) and $\mathbb{Y}^{p,q,p_1,2} = W^{1,p}\left(0,T;\mathbb{Y}^{q,p_1,2},\mathbb{X}_{p,q,p_1}\right)$

Theorem 5.2. Assume thar the second assumption of Condition 2.1 is satisfied. Moreover, let $b_1 \in VMO \cap L^{\infty}(0,1)$, $\operatorname{Re}\omega_k \neq 0$, $\frac{\lambda}{\omega_k} \in S(\phi_1)$ for a.e. $x \in (0,1)$, $\phi_1 \in [0\ \pi)$ and $b_0 \in VMO \cap L^{\infty}(0,1)$, $b_1(0) = b_1(1)$, $b_0(0) = b_0(1)$. Then for every $f \in \mathbb{X}_{p,q,p_1}$, $a \in \mathbb{V}_{p,q}$, $p,q,p_1 \in (1,\infty)$ there is a unique solution $u \in \mathbb{Y}^{p,q,p_1,2}$ and $\varphi \in H^{1,p}(\mathbb{R}^n; L^{p_1}(0,1))$ of the problem (1.14)-(1.15). Moreover the following coercive estimate holds

$$\left\|\frac{\partial u}{\partial t}\right\|_{\mathbb{X}_{p,q,p_1}} + \sum_{k=1}^{n}\left\|\frac{\partial^2 u}{\partial x_k^2}\right\|_{\mathbb{X}_{p,q,p_1}} + \|A_1 u\|_{\mathbb{X}_{p,q,p_1}} + \tag{5.2}$$

$$\|\nabla\varphi\|_{\mathbb{X}_{p,q,p_1}} \lesssim \|f\|_{\mathbb{X}_{p,q,p_1}} + \|a\|_{\mathbb{V}_{p,q}}.$$

Proof. By [14, Theorem 5.0] the operator A_1 is R-positive in $L^{p_1}(0,1)$. Therefore, all conditions of Theorem 4.1 are hold and we obtain the conclusion.

Data Availibility: Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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